Analysis 1

10 January 2024

Warm-up: If $u = \sin(x)$, then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = ? \qquad \mathrm{d}u = ? \qquad \mathrm{d}x = ?$$

Last time

Indefinite integrals

$\int f(x)\,\mathrm{d}x$ describes all the anti-derivatives of a function.

Example: $\int \sin(3x)\,\mathrm{d}x = \frac{-1}{3}\cos(3x) + C$

Definite integrals

$\int_a^b f(x)\,\mathrm{d}x$ is the “signed area” under the graph $y = f(x)$ from $x = a$ to b.

Instead of using area formulas, we often calculate this using the FTC.

Last time

The Fundamental Theorem of Calculus

If f is continuous then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a),$$

where $F(x)$ is any function for which $F'(x) = f(x)$.

The subtraction on the right can also be written $F(x)\Bigg|_{x=a}^{x=b}$ or $\Bigg[F(x)\Bigg]_{x=a}^{x=b}$.

Last time

Example: $\int_{-5}^{4} \frac{1}{3}x^2\,\mathrm{d}x = \frac{1}{9}x^3\Big|_{x=-5}^{x=4}$

$= \frac{1}{9}(4)^3 - \frac{1}{9}(-5)^3$ ← This is F(b) - F(a).

$= \frac{64}{9} - \frac{-125}{9}$

$= \frac{189}{9}$

$= 21$

u-substitution

Last time

When we see a function and its derivative in a certain configuration, we can re-write an integral using "substitution".

- As a general formula, we have

  $$\int f\big(g(x)\big)\,g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u, \qquad \text{where } u = g(x).$$

  but examples may be easier to understand than this formula.

- We often use u as the new variable of integration, so this method is also called "u-substitution".

Last time

$$\int (x^2+5)^8\,(2x)\,dx. = \int u^8\,du = \frac{1}{9}u^9 + C = \boxed{\frac{1}{9}(x^2+5)^9 + C}$$

$$u = x^2 + 5$$

$$\frac{du}{dx} = 2x$$

$$du = 2x\,dx$$

Task 1: Find $\displaystyle\int \frac{24x}{\sqrt{4x^2-1}}\,\mathrm{d}x$.

Using $u = 4x^2 - 1$, ...

Answer: $6\sqrt{4x^2-1} + C$

Task 2: Calculate $\int_{1/2}^{(\sqrt{5})/2} \frac{24x}{\sqrt{4x^2-1}}\,\mathrm{d}x$..

FTC

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

where $F'(x) = f(x)$.

Option 1: Using the final answer to the previous task, this is $6\sqrt{4x^2-1}\Big|_{x=1/2}^{x=(\sqrt{5})/2} = \ldots$

Option 2: Change x-values into u-values at top and bottom of integral: $\int_0^4 3u^{-1/2}du = 6u^{1/2}\Big|_{u=0}^{u=4} = 12.$

Task 3: Find $\int_0^1 2(x^2+3)^2\,\mathrm{d}x$ $= \int_0^1 (2x^4 + 12x^2 + 18)dx$

Answer: 22.4

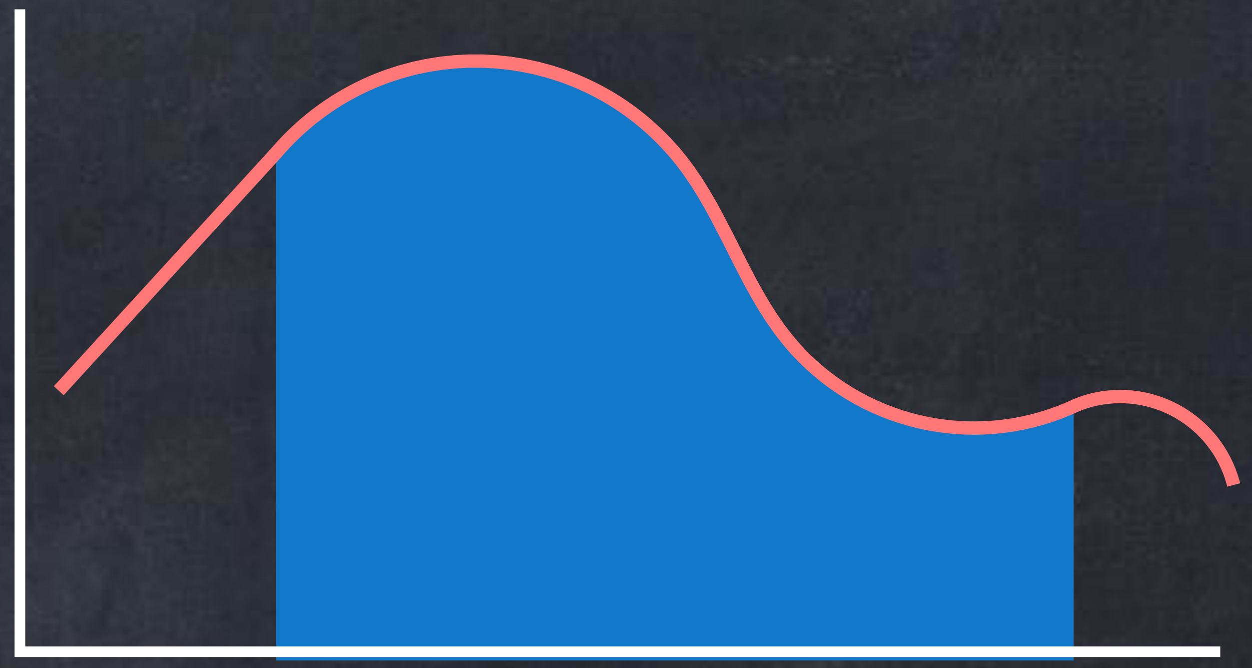
NOT U-SUB!

Task 4: Evaluate $\int_1^7 \frac{x}{x^2+1}\mathrm{d}x$.

Answer: $\frac{1}{2}\ln(25)$, or $\ln(5)$

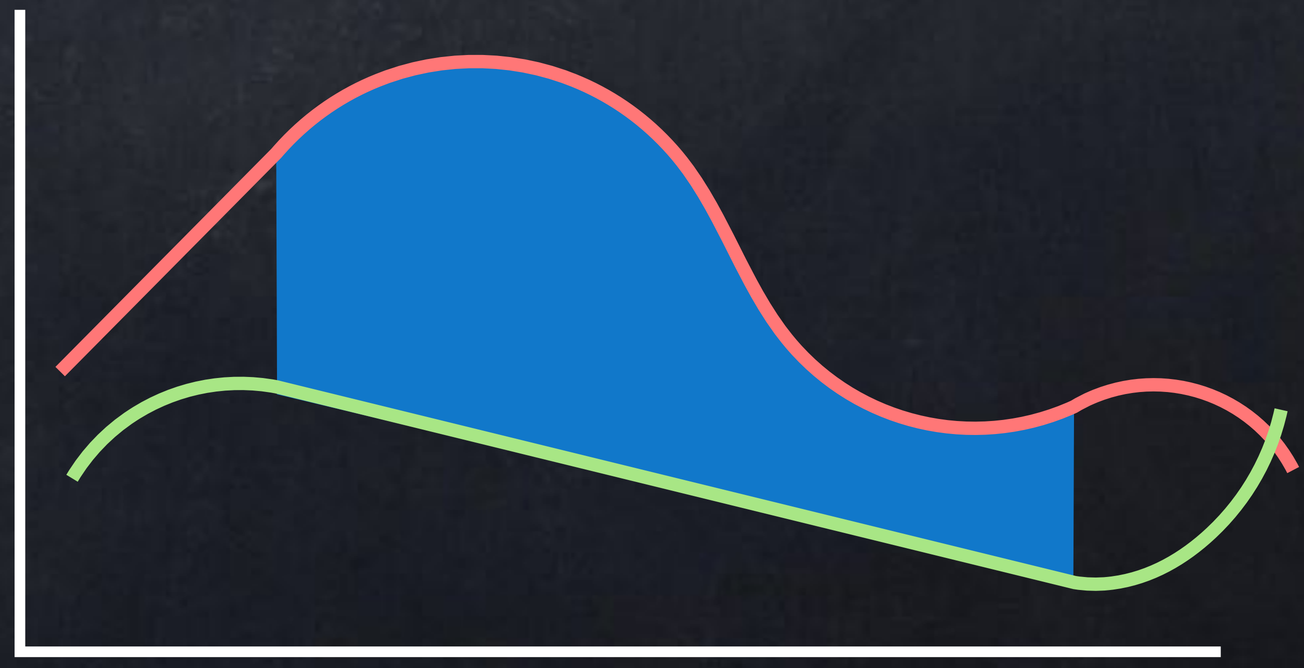
Area between curves

Previously we defined $\int_a^b f(x)\mathrm{d}x$ as the area between $y = f(x)$ and the x-axis with $a \leq x \leq b$.

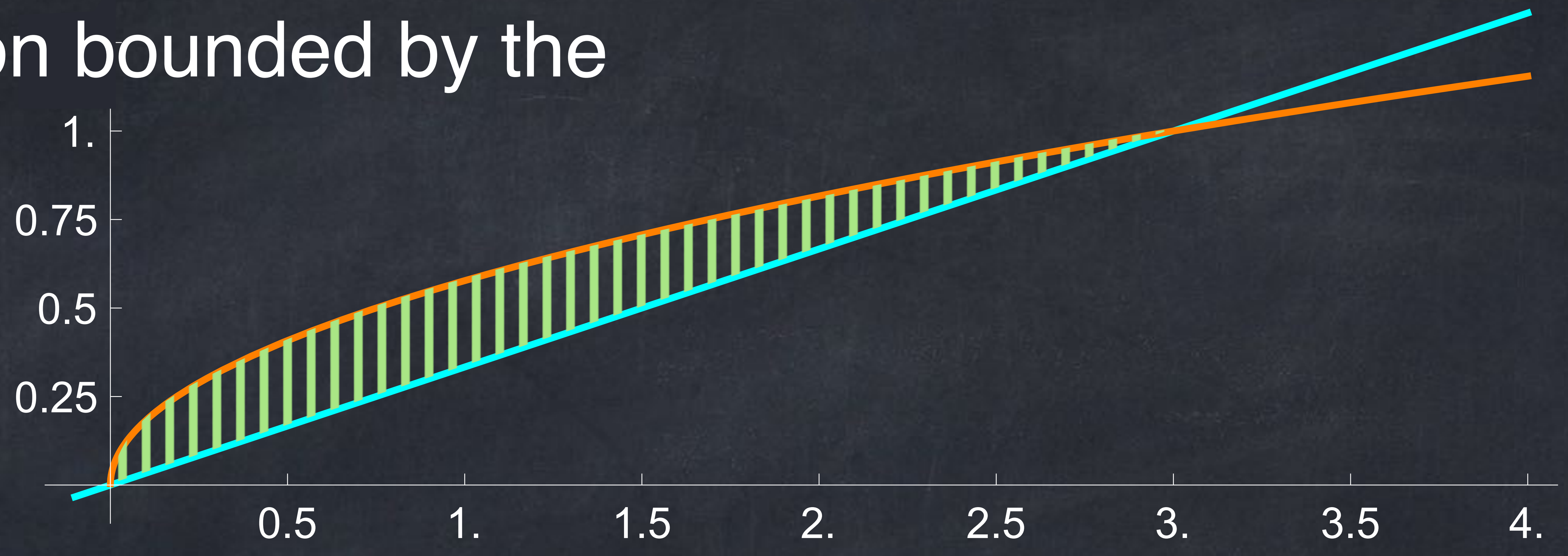


If $f(x) \leq g(x)$ for all $a \leq x \leq b$ then the area between the curves $y = f(x)$ and $y = g(x)$ with $a \leq x \leq b$ is

$$\int_a^b \big(g(x) - f(x)\big)\mathrm{d}x.$$



Example: Find the area of the region bounded by the curves $\underbrace{y = \frac{x}{3}}_{\text{bottom}}$ and $\underbrace{x = 3y^2}_{\text{top}}$.



$$\text{Area} = \int_{\text{left}}^{\text{right}} (\text{top} - \text{bottom})\,dx = \int_0^3 \left(\frac{1}{\sqrt{3}}x^{1/2} - \frac{1}{3}x\right)dx$$

$$= \ldots = \frac{1}{2}$$

Area between curves

The area of a shape with $a \leq x \leq b$ and with curves on the top and bottom is

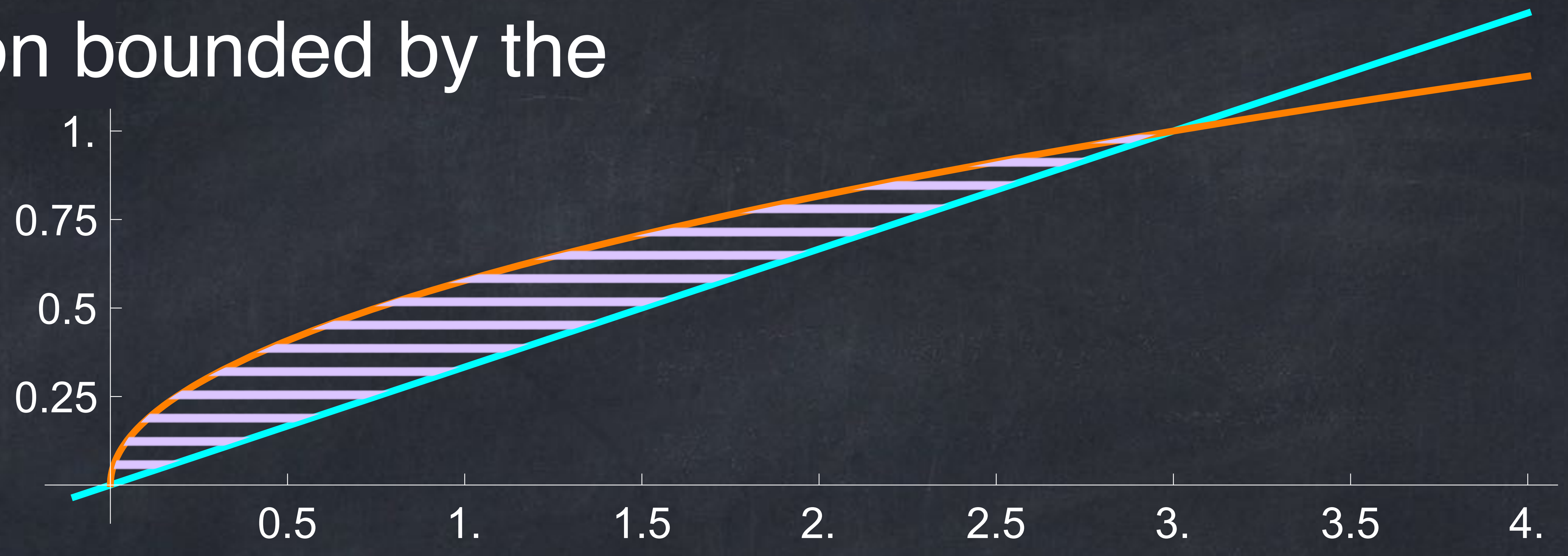
$$\int_a^b \big(\mathrm{Top}(x) - \mathrm{Bottom}(x)\big)\mathrm{d}x.$$

The area of a shape with $c \leq y \leq d$ and with curves on the left and right is

$$\int_c^d \big(\mathrm{Right}(y) - \mathrm{Left}(y)\big)\mathrm{d}y.$$

For some shapes, both methods are possible!

Example: Find the area of the region bounded by the curves $\underbrace{y = \frac{x}{3}}_{\text{right}}$ and $\underbrace{x = 3y^2}_{\text{left}}$.



$$\text{Area} = \int_{\text{bottom}}^{\text{top}} (\text{right} - \text{left})\,dx = \int_0^1 \left(3y - 3y^2\right) dy$$

$$= \ldots = \frac{1}{2}$$